Introduction to the Linear Regulator

MEM 355 Performance Enhancement of Dynamical Systems

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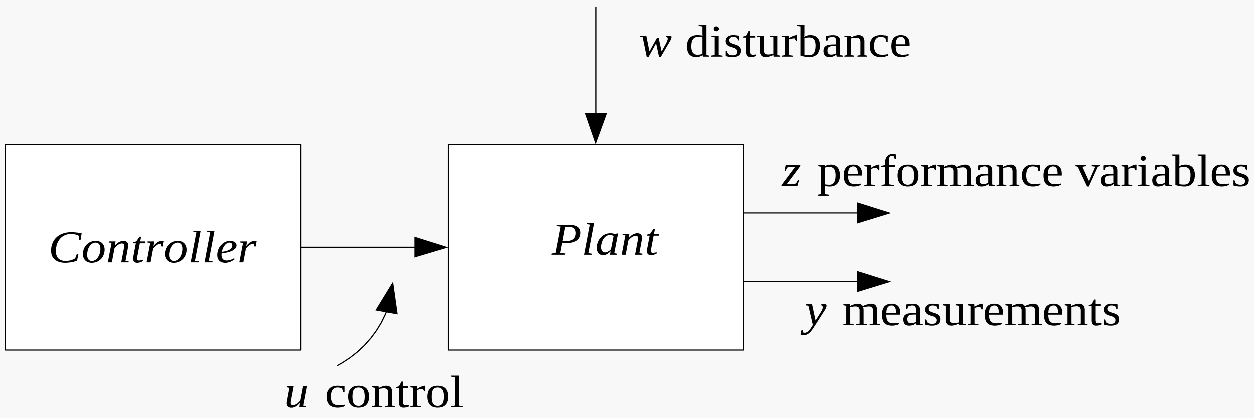
Disturbance Rejection Setup



state equations	$\dot{x} = Ax + Ew + Bu$
disturbance	$\dot{w} = Zw$
performance	$z = Cx + Fw + Du$
measurements	$y = \bar{C}x + \bar{F}w + \bar{D}u$

Eigenvalues of Z ordinarily on Im axis

In effect, the error

$$x \in R^n, w \in R^q, u \in R^m, z \in R^m, y \in R^p$$

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Objectives

Regulator Problem: Find a controller to achieve the following

1) Regulation: $z(t) \to 0$ as $t \to \infty$

2) (Internal) Stability: Achieve specified transient response

Robust Regulator Problem: Find a solution to the Regulator Problem that satisfies

3) Robustness: 1) and 2) should be maintained under specified small perturbations of plant and/or control parameters

Solution: Part 1- Regulation

Consider the possibility of a control $\bar{u}(t)$ that produces a trajectory $\bar{x}(t)$ for some unspecified initial state $\bar{x}_0$ and any initial disturbance vector w_0, so that the corresponding $\bar{z}(t) \equiv 0$. Then, $\bar{x}, \bar{u}, w$ must satisfy

$$\dot{\bar{x}} = A\bar{x} + Ew + B\bar{u}$$
$$\dot{w} = Zw$$
$$0 = C\bar{x} + Fw + D\bar{u}$$

Assume a solution of the form: $\bar{x} = Xw, \bar{u} = Uw \Rightarrow$

$$\left.\begin{aligned} XZw &= AXw + Ew + BUw \\ 0 &= CXw + Fw + DUw \end{aligned}\right. \quad \forall w$$

Thus, the hypothesized control $\bar{u}(t)$ exists if there are X, U that satisfy

$$\boxed{\begin{aligned} -XZ + AX + BU &= -E \\ CX + DU &= -F \end{aligned}}$$

Solutions typically are not unique

Solution: Part 2- Stability

Define $\delta u, \delta x$

$$\left.\begin{aligned} u &= \bar{u} + \delta u = Uw + \delta u \\ x &= \bar{x} + \delta x = Xw + \delta x \end{aligned}\right. \Rightarrow \delta\dot{x} = A\delta x + B\delta u$$

Now, if (A, B) is controllable, it easy to choose $\delta u = K\delta x$

so that the closed loop $\delta\dot{x} = (A + BK)\delta x$ has desired transient

characteristics.

With K chosen, the control can be written as a function of the system

states x, w

$$u = \bar{u} + \delta u = Uw + \delta u = Uw + K\delta x \Rightarrow$$

$$\boxed{u = Uw + K(x - Xw) = \begin{bmatrix} K & U - KX \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix} = K_{TOT}\begin{bmatrix} x \\ w \end{bmatrix}}$$

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Solution: Part 3- Observation

The control will be implemented using estimates of the composite state (x, w). Consider the composite system

$$\frac{d}{dt}\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} A & E \\ 0 & Z \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$$y = \bar{C}x + \bar{F}w + \bar{D}u$$

If the composite system is observable, we can choose a matrix L, so that the following observer has the desired dynamics:

$$\frac{d}{dt}\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} = \begin{bmatrix} A & E \\ 0 & Z \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + L\left(\bar{C}\hat{x} + \bar{F}\hat{w} + \bar{D}u - y\right)$$

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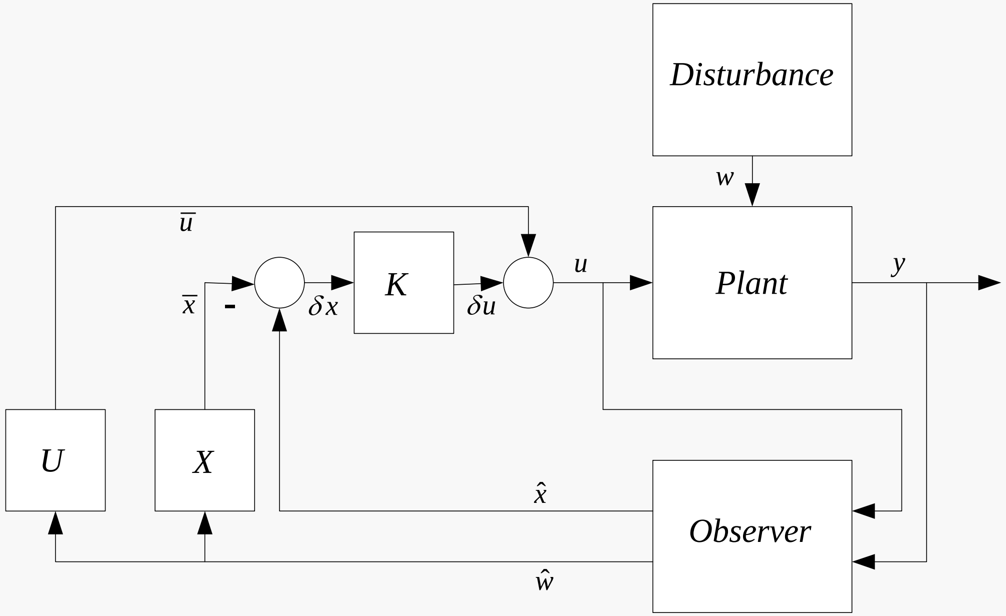
Properties of the Loop



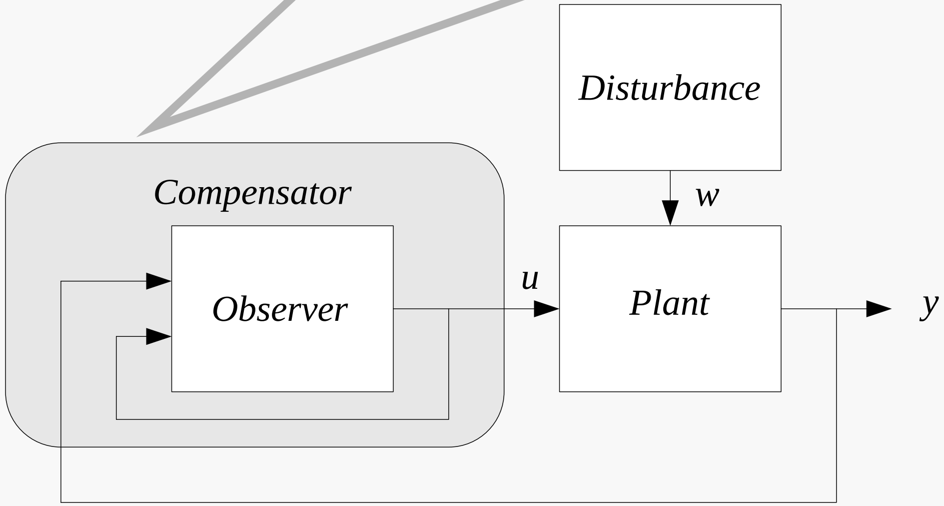
$$\begin{bmatrix}\dot{\hat{x}}\\ \dot{\hat{w}}\end{bmatrix} = \left(\begin{bmatrix}A & E\\ 0 & Z\end{bmatrix} + L\begin{bmatrix}\bar{C} & \bar{F}\end{bmatrix}\right)\begin{bmatrix}\hat{x}\\ \hat{w}\end{bmatrix} + \left(\begin{bmatrix}B\\ 0\end{bmatrix} + L\bar{D}\right)u - Ly$$

$$u = \begin{bmatrix}K & U-KX\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{w}\end{bmatrix}$$

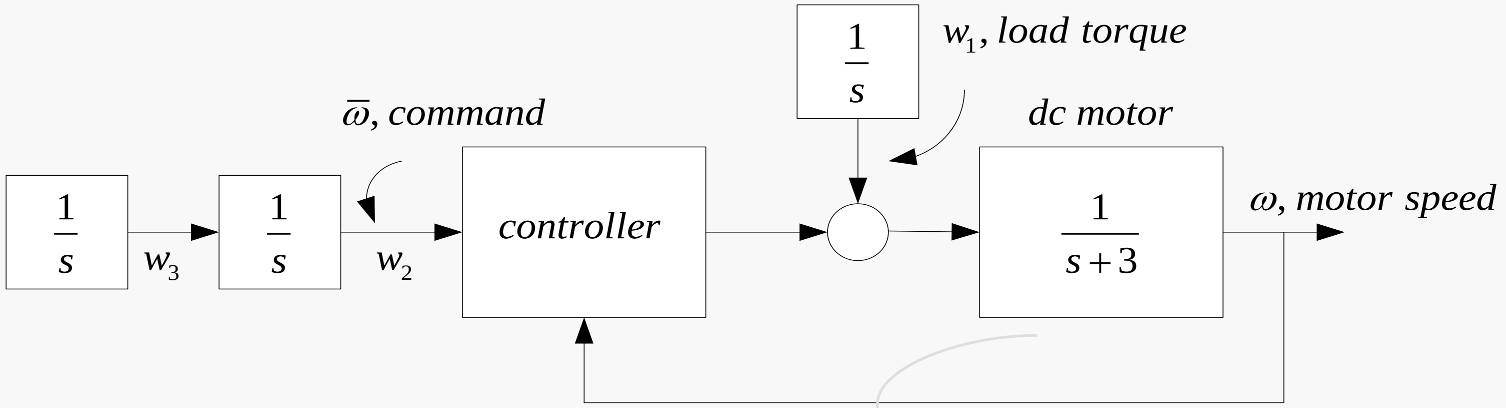
$$\begin{bmatrix}\dot{\hat{x}}\\ \dot{\hat{w}}\end{bmatrix} = \left(\begin{bmatrix}A & E\\ 0 & Z\end{bmatrix} + L\begin{bmatrix}\bar{C} & \bar{F}\end{bmatrix} + \left(\begin{bmatrix}B\\ 0\end{bmatrix} + L\bar{D}\right)\begin{bmatrix}K & U-KX\end{bmatrix}\right)\begin{bmatrix}\hat{x}\\ \hat{w}\end{bmatrix} - Ly$$

$$u = \begin{bmatrix}K & U-KX\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{w}\end{bmatrix}$$

Compensator contains copy of *Z* ~ internal model of disturbance



Example



$$\dot{x} = -3x + w_1 + u \quad \left(\dot{\omega} = -3x + w_1 + u\right)$$

Plant

$$\frac{d}{dt}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

Disturbance

Performance

$$y = \begin{bmatrix} \omega \\ \bar{\omega} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \quad z = \bar{\omega} - \omega = -x + \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

Measurements

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Example, Cont'd

Step 1 (Regulation): $X = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} -1 & 4 & 0 \end{bmatrix}$

Step 2 (Stabilization): $k = -2 \Rightarrow \lambda = -5$

Step 3 (Observation): note 2 decoupled parts

$\omega \to \hat{x}, \hat{w}_1$

$A = \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$L = \begin{bmatrix} -19 \\ -120 \end{bmatrix} \Rightarrow \lambda_{1,2} = -10, -12$

$\bar{\omega} \to \hat{w}_2, \hat{w}_3$

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$L = \begin{bmatrix} -22 \\ -120 \end{bmatrix} \Rightarrow \lambda_{1,2} = -10, -12$

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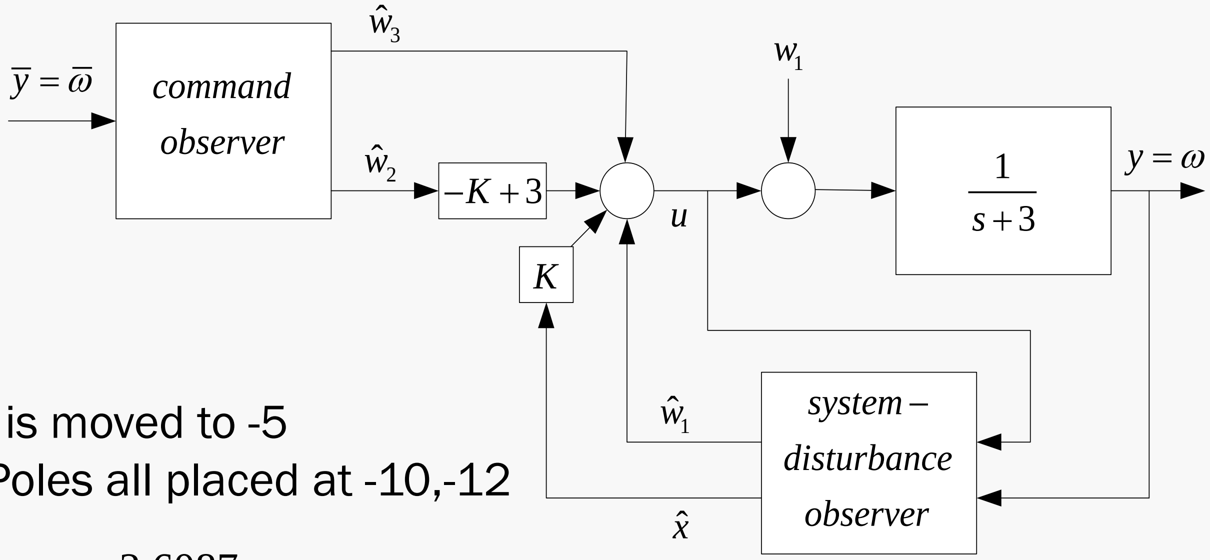
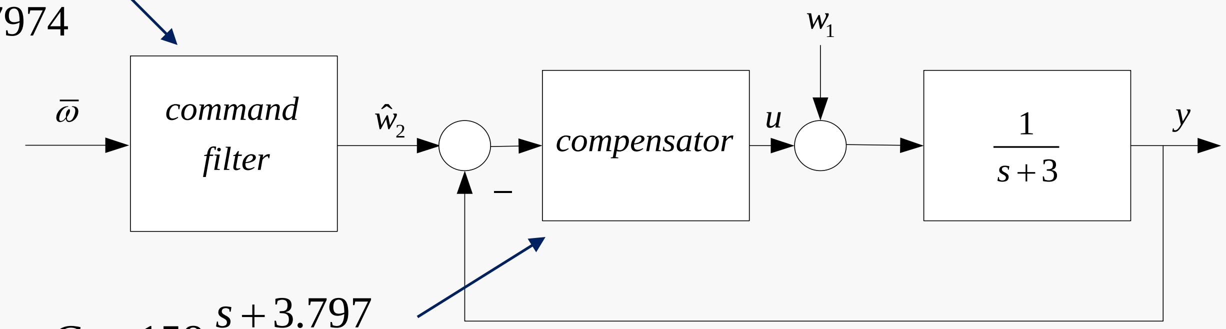
Example, Cont’d



Plant pole is moved to -5

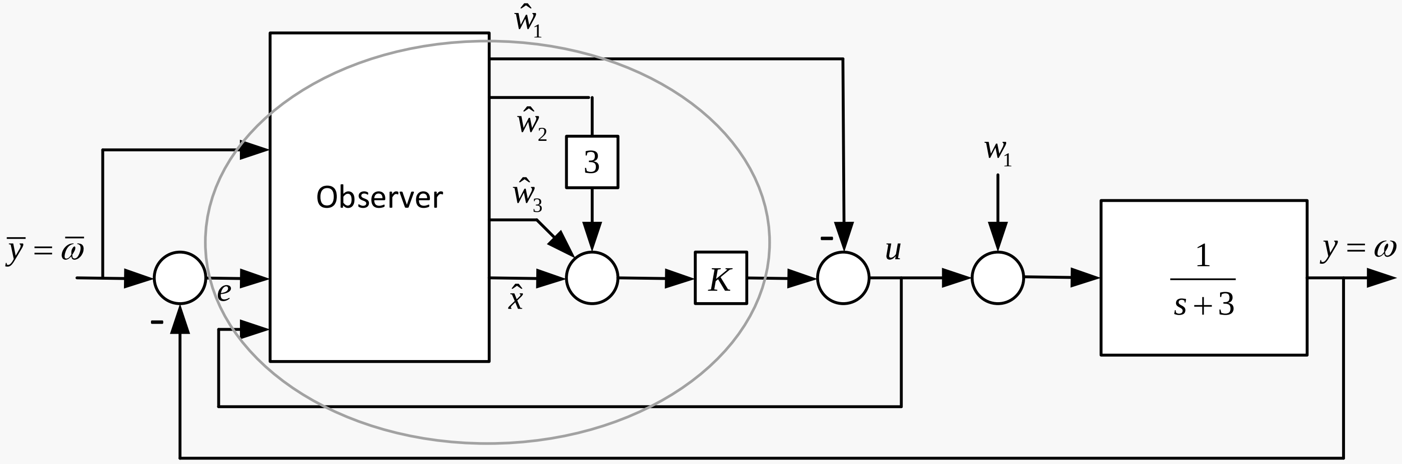
Observer Poles all placed at -10,-12

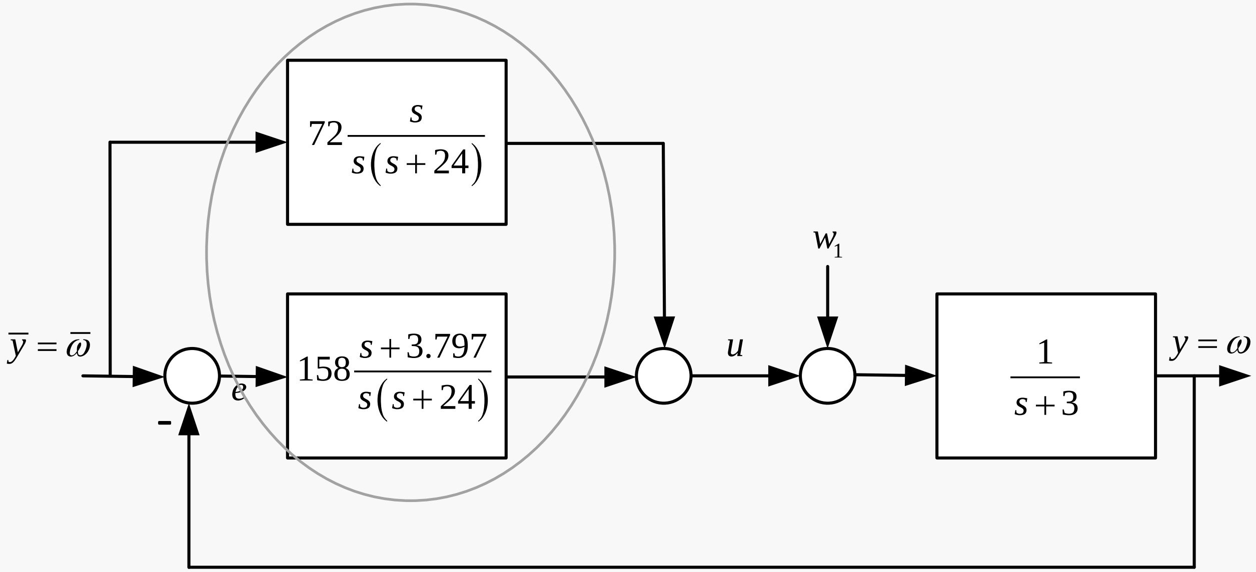
$$1.4557\frac{s+2.6087}{s+3.7974}$$



$$G_c = 158\frac{s+3.797}{s(s+24)}$$

Example Cont'd





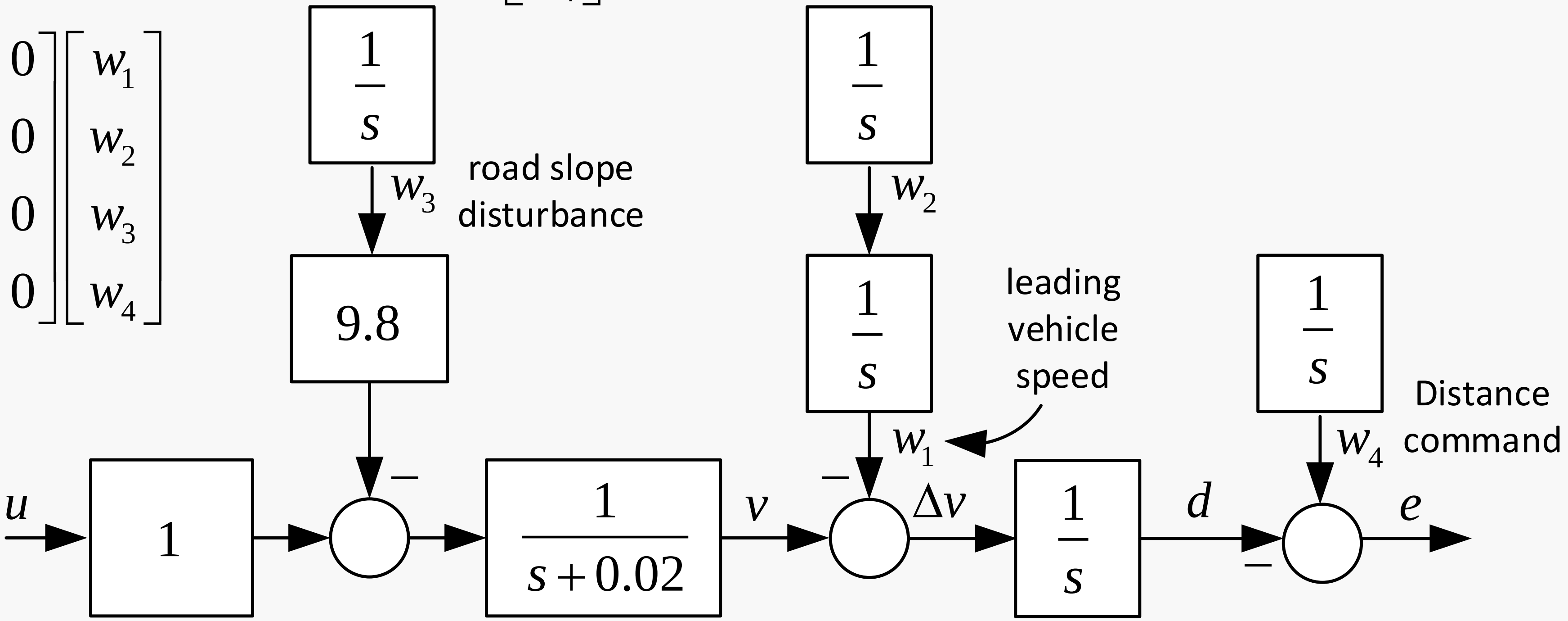
Cruise Control Example: Distance Following Mode

$$\frac{d}{dt}\begin{bmatrix} d \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.02 \end{bmatrix}\begin{bmatrix} d \\ v \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -9.8 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\frac{d}{dt}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

$$e = -d + w_4,$$

$$y = \begin{bmatrix} d & v & w_4 \end{bmatrix}^T$$



Cruise Control, 2

$$-XZ + AX + BU + E = 0$$

$$CX + DU + F = 0$$

$$-\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & -0.02 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -9.8 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix} = 0$$

$$\Rightarrow X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0.02 & 1 & 9.8 & 0 \end{bmatrix}$$

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Cruise Control, 3

For controller poles: -5,-4 yields $K = -\begin{bmatrix} 20 & 8.98 \end{bmatrix} \Rightarrow A + BK = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix}$

For observer:

```
>> Ae=[0,1,-1,0,0,0;0,-0.02,0,0,-.98,0;
        0,0,0,1,0,0;0,0,0,0,0,0;
        0,0,0,0,0,0;0,0,0,0,0,0];
>> Ce=[1,0,0,0,0,0;0,1,0,0,0,0;0,0,0,0,0,1];
>> poles=[-20,-21,-22,-23,-24,-25];
>> place(Ae',Ce',poles)'
ans =
   1.0e+04 *
    0.0072    0.0001         0
    0.0000    0.0043         0
   -0.1727   -0.0001         0
   -1.3800   -0.0015         0
   -0.0000   -0.0471         0
         0         0    0.0020
```

Cruise Control